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ON THE BEST METHODS FOR TIME DOMAIN ANALYSIS OF RF CIRCUITS

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Abstract. Currently employed time-domain simulation methods, which are based on trapezoidal rule (TR), or backward differentiation formulas (BDF), reveal to be both poorly accurate and inefficient when employed to simulate oscillating circuits. Only L-stable methods can solve stiff systems of ordinary differential equations (SODE) of most practical circuits and only P-stable methods can solve SODE of oscillation circuits. A trapezoidal rule is P-stable, but isn't L-stable. BDFs are L-stable, but do not have P-stability. This paper presents a one-step multi-stage backward differentiation formula (TR-BDF4) of a second-order, as a generalization of the TR-BDF composite scheme. This scheme is equivalent to singly diagonally implicit Runge-Kutta (SDIRK) methods of special type, and regarded as the one-step analogs of the multi-step methods, the backward differentiation formulas (BDFs). TR-BDF4' method is A(π/2)-stable, is not P-stable, but it has an interval of periodicity (0; 0.37). Unlike the conventional BDFs and TR-BDF the TR-BDF4 at small steps maintains the accuracy of the trapezoidal formula at the imaginary axis.

It is further shown that the implicit difference schemes solvable by Gauss-Seidel iterations have some similar properties.

Keywords: oscillating circuits, nonlinear differential equations, error analysis.

1. Introduction

High-performance circuit simulation has to deal with a lot of challenges for which at this moment several pragmatic 'engineering' approaches are used and where a sound mathematical background is lacking. In fact, simulation has to become orders faster and has to deal with circuits that are orders larger than met before [1]. The

design of the radio frequency (RF) section in a communication integrated circuit (IC) is a challenging problem. Although several computer-aided analysis tools are available for RFIC design, they are not effectively used, because there is a lack of understanding about their features and limitations.

Currently employed time-domain simulation methods, which are based on trapezoidal rule (TR), or backward differentiation formulas (BDF), reveal to be both poorly accurate and inefficient when employed to simulate oscillating circuits. Only L-stable methods can solve stiff systems of ordinary differential equations (SODE) of most practical circuits and only P-stable methods can solve SODE of oscillation circuits. Most SPICE-like simulators for the circuits' analysis in the time domain use methods of trapezoids and BDF. A trapezoidal rule is P-stable, but isn't L-stable. BDFs are L-stable, but do not have P-stability.

Improvements of BDF were looked for in combining BDF with the Trapezoidal Rule [2, 3]. TR-BDF2 method is more stiffly accurate, it's stepsize control is more effective, but damping behavior along the imaginary axis is still insufficient.

In this paper, we show that composite implicit method TR-BDF2 can be extended to a one-step multi-stage scheme, where each stage uses a backward differentiation formula, TR-BDF4 [4]. Here, the TR-BDF4 scheme is analogous to the standard BDF with 4-steps.

In this paper, we present TR-BDF4 scheme and relate it to TR-BDF2. The stability properties of these two second-order singly diagonally implicit Runge-Kutta (SDIRK2) schemes are analyzed and compared with an original BDF2 and trapezoid rule. Comparison of properties of methods carried out separately for two test problems that have the eigenvalues of the Jacobi matrix are located either on the real or imaginary axes.

Next, we show that the implicit scheme remains in some way a stiff-stable, but becomes undamped at a part of imaginary axis when it's solved by symmetric Gauss-Seidel iterations [5, 6].

2. The second-order TR-BDF2 and TR schemes

Assume $f(x)$ is a sufficiently smooth nonlinear function of x. We consider

numerically integrating the following autonomous ordinary differential equation,

$$dx/dt = f(x), \tag{1}$$

from time t_n to $t_{n+1} = t_n + h$. The extension of the work to non-autonomous ODEs is standard and straightforward.

Our discussion starts with the following two schemes:

• *The trapezoidal scheme* (*TR*),

$$x_{n+1} = x_n + (h/2)(f_n + f_{n+1}). \tag{2}$$

• *The implicit trapezoidal and BDF2 composite scheme* (*TR-BDF2*),

$$x_{n+1/2} = x_n + (h/4)(f_n + f_{n+1/2}), \tag{3a}$$

$$x_{n+1} = (4/3)x_{n+1/2} - (1/3)x_n + (2/3)hf_{n+1}. \tag{3b}$$

The TR-BDF2 scheme (3) was originally derived as a composite method of the trapezoidal rule and the backward-differentiation-formula of second-order (BDF2) [2, 3]. Unlike the standard BDFs, however, the composite BDFs do not need external startup calculation while maintaining the full accuracy of the scheme. By replacing the trapezoidal part in the TR-BDF2 scheme with the second-order implicit midpoint rule, another composite scheme can be derived with the same A- and L-stabilities as TR-BDF2 [7]. The leading order term of the local truncation error of a second-order scheme is $E_T \propto Ch^3$. For method (3) the proportionality constant is $C_{\text{TR-BDF2}} = -0.0404$, for its parent methods the error constants are $C_{\text{TR}} = -0.0833$, and $C_{\text{BDF2}} = -0.222$.

3. TR-BDF3 and TR-BDF4 methods

Now we present the generalization of the second-order composite scheme TR-BDF3 that is one-step modification of 3-d order BDF scheme [6]

$$x_{n+1/3} = x_n + (h/6)(f_n + f_{n+1/3}), \tag{4a}$$

$$x_{n+2/3} = (4/3)x_{n+1/4} - (1/3)x_{.n} + (4/9)hf_{n+2/3}, \tag{4b}$$

$$x_{n+1} = (18/11)x_{n+2/3} - (9/11)x_{n+1/3} + (2/11)x_n + (2/11)hf_{n+1}. \tag{4d}$$

In the same way, we obtain the BDF4 method

$$x_{n+1/4} = x_n + (h/8)\left(f_n + f_{n+1/4}\right), \tag{5a}$$

$$x_{n+1/2} = (4/3)x_{n+1/4} - (1/3)x_{.n} + (1/6)hf_{n+1/2}, \tag{5b}$$

$$x_{n+3/4} = (18/11)x_{n+1/2} - (9/11)x_{n+1/4} + (2/11)x_n + (3/22)hf_{n+3/4} \tag{5c}$$

$$x_{n+1} = (48/25)x_{n+3/4} - (36/25)x_{n+1/2} + (16/25)x_{n+1/4} - (3/25)x_n + (3/25)hf_{n+1}. \tag{5d}$$

All TR-BDFs don't need external startup calculation. Note, BDF3 and BDF4 are not A-stable, however TR-BDF3,4 remains A-stable as the TR-BDF2. The proportionality factors of the leading order term of the local truncation error of TR-BDF3 and TR-BDF4 are $C_{\text{TR-BDF3}} = -0.0177$, and $C_{\text{TR-BDF4}} = -0.00765$. The main difference of TR-BDF4 is the fact that it does not have damping at $|\omega h| \le 0.36$ along the imaginary axis. As a consequence, the TR-BDF4-method is more suited for oscillatory problems than the TR-BDF2,3-methods. All the variables in (5) are calculated separately, that is, TR-BDF4 is SDIRK method [8].

4. Seidel iterations

The classical univariate time-step integration (SPICE-like simulation) is known to be inefficient on both computational time and memory storage for VLSI. The main disadvantage of an implicit difference schemes is the quadratic dependence of the CPU time on the SODE dimension *m* for Newton iterations. The 'time advance' algorithm, using symmetric Seidel's iterations, allows this dependence to be linear [5]. The regions of absolute stability of implicit methods, when the symmetric Seidel iterations are implemented, usually partially coincide with the imaginary axis [6], so that at small steps they have P-stability.

For the implicit trapezoid method (2), symmetric Seidel iterations for *m*-dimensional SODE (TS) are given by two half-steps (6a) and (6b). The basic idea here is to 'symmetrize' the Gauss-Seidel scheme with a method that takes two half steps of size *h*/2 each: one half step is taken in the usual 'forward' direction, the second half step in the backward direction.

$$x_{k,n+1/2} = x_{k,n} + (h/4) f_k(x_{1,n+1/2}, \ldots x_{k,n+1/2}, x_{k+1,n}, \ldots, x_{m,n}\ t_{n+1/2}),$$

$$+ (h/4) f_k(x_{1,n}, \ldots, x_{k,n}, x_{k+1,n}, \ldots, x_{m,n},\ t_n) \qquad k = 1, 2, \ldots, m, \quad (6a)$$

$$x_{k,n+1} = x_{k,n+1/2} + (h/4) f_k(x_{1,n+1/2}, \ldots,\ x_{k-1,n+1/2}, x_{k,n+1}, \ldots, x_{m,n+1},\ t_{n+1}),$$

$$+ (h/4) f_k(x_{1,n+1/2}, \ldots, x_{k-1,n+1/2}, \ldots, x_{k,n+1/2}, \ldots,\ x_{m,n+1/2},\ t_{n+1/2}),$$

$$k = m,\ m-1, \ldots 1. \quad (6b)$$

As method (2) isn't L-stable, the scheme (6) has no L-stability too. To obtain L-stability we'll use two-stage fully implicit method Runge-Kutta of 2-nd order

$$x_{n+1} = x_n + h\ f\ [x_{n+1} - 0.5\, h\ f\,(x_{n+1},\ t_{n+1}),\ t_n + 0.5\, h]\ .$$

For Seidel symmetric iterations we get the set of decoupled equations (7) (RKS).

$$\begin{cases} x_{k,n+1/2} = x_{k,n} + (h/2) f_k\left(x_{1,n+1/4}, x_{2,n+1/4}, \ldots, x_{k,n+1/4}, x_{k+1,n}, \ldots, x_{m,n}, t_{n+1/4}\right), \\ x_{k,n+1/2} = x_{k,n+1/4} + (h/4) f_k\left(x_{1,n+1/2}, x_{2,n+1/2}, \ldots, x_{k,n+1/2}, x_{k+1,n}, \ldots, x_{m,n}, t_{n+1/2}\right); \end{cases}$$

$$k = 1, 2, \ldots, m, \quad (7a)$$

$$\begin{cases} x_{k,n+1} = x_{k,n+1/2} + (h/2) f_k\left(x_{1,n+1/2}, x_{2,n+1/2}, \ldots, x_{k-1,n+1/2}, x_{k,n+3/4}, \ldots, x_{m,n+3/4}, t_{n+3/4}\right), \\ x_{k,n+1} = x_{k,n+1/2} + (h/4) f_k\left(x_{1,n+1/2}, x_{2,n+1/2}, \ldots, x_{k-1,n+1/2}, x_{k,n+1}, \ldots, x_{m,n+1}, t_{n+1}\right); \end{cases}$$

$$k = m, m-1, \ldots, 1\ . \quad (7b)$$

5. Numerical examples

5.1. Example 1

A fundamental numerical problem that severely limits the usefulness of many computer simulation programs is the time-constant problem associated with stiff differential equations. Stiffness occurs in a problem where there are two or more very different scales of the independent variable on which the dependent variables are changing. For example, consider the following set of equations [9]

$$u' = 998\, u + 1988\, v,\ v' = -999\, u - 1999\, v. \quad (8)$$

With initial conditions $u(0) = 1$, $v(0) = 0$ the solution is

$$u = 2e^{-t/\tau_1} - e^{-t/\tau_2}, v = -e^{-t/\tau_1} + e^{-t/\tau_2}.$$

Here $\tau_1 = 1$ and $\tau_2 = 0.001$ are time constants. If we integrated the system (8) with any of the methods given above, the asymptotic slope, at $h \to 0$, of a global error accumulated at the end of the time interval of interest as a function of stepsize in logarithmic scales is equal to scheme's order, as Fig. 1 shows.

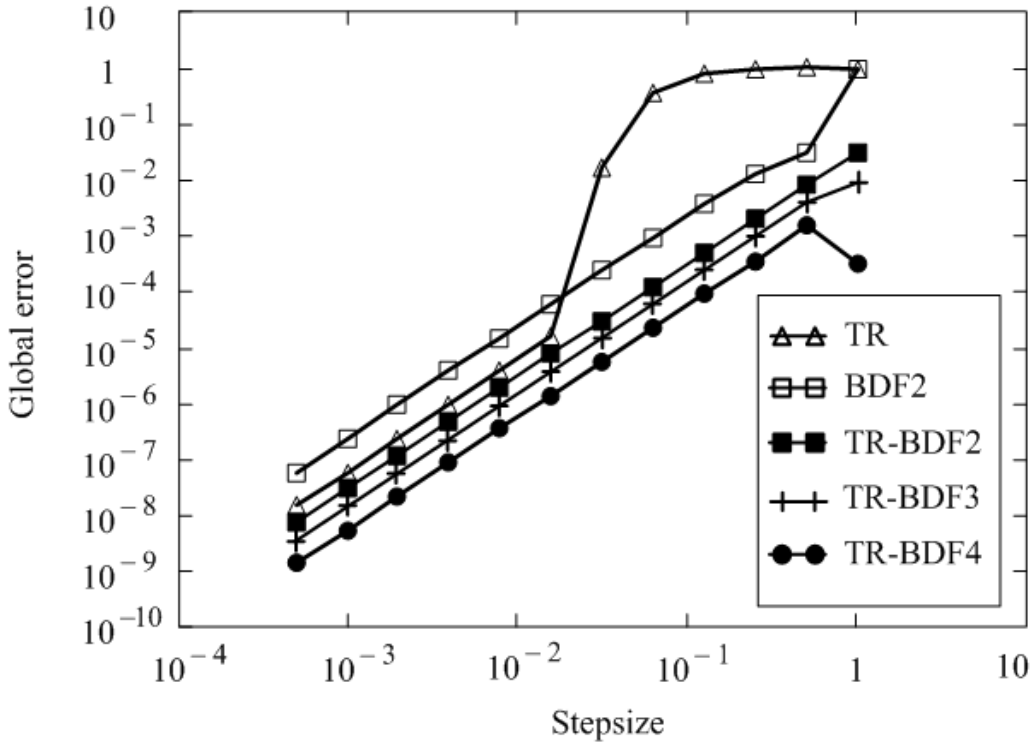

Fig. 1. Global error $\varepsilon = |v_n - v(t_n)|$ at $t_n = 1$ of (1) – (4) methods applied with various stepsizes h to the test problem (5). The power law exponent for $h \to 0$ is 2.

L-stable methods in logarithmic scales have linear dependence $\varepsilon(h)$ up to $h \approx \tau_{\max}$. The trapezoidal scheme has no L-stability, so $\varepsilon_{TR}(h)$ rapidly rises to maximum at $h \geq 10\tau_{\min}$.

5.2. Example 2

Unfortunately, as Fig 3 shows, the solution of (8) by the symmetric iterations of the Seidel shows that method (6) does not have L-stability in this case.

The efficiency of methods (6) and (7) depends essentially on the magnitude of the diagonal elements of the Jacobi matrix. For large diagonal elements, the methods

have A(α)-stability, for zero coefficients they become conditionally stable [6]. Method (6) was developed specifically for *n*-channel MOS VLSI logic, that are non-linear *RC*-circuits without feedbacks. The scheme of the simplest *RC*-circuit of second order is shown in Fig. 2. System of equations of such circuits have diagonally dominant Jacobi matrix. As an example of such equations, we consider the system of equations in which the unknowns are capacitors' voltages

$$v_{C1}' = 1000\, v_{C1} - 1000\, v_{C2}; \quad v_{C1}' = 2v_{C1} - v_{C2}. \tag{9}$$

With initial conditions $v_{C1}(0) = 1$, $v_{C2}(0) = 0$ the solution is

$$v_{C1} = 2\exp(-t/\tau_1) - \exp(-t/\tau_2),\ v_{C2} = \exp(-t/\tau_2) - \exp(-t/\tau_1).$$

Solution of (9) has the same stiffness τ_1 / τ_2 as (8).

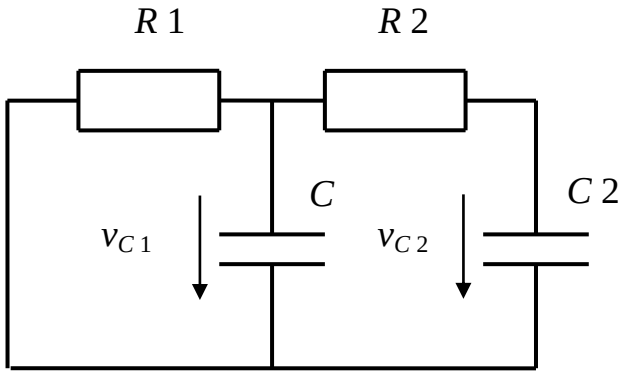

Fig. 2. The *RC*-scheme.

The corresponding functions of $\varepsilon = \left| v_{C2,n} - v_{C2}(t_n) \right|$, $t_n = 1$ for (8) and (9) are shown in Fig. 3.

For comparison, the same figure shows similar results for the implicit Euler method (ES). The Seidel iteration scheme for the implicit Euler method has the form of

$$x_{k,n+1/2} = x_{k,n} + (h/2) f_k(x_{1,n+1/2}, x_{2,n+1/2}, \ldots, x_{k,n+1/2}, x_{k+1,n}, \ldots, x_{m,n}, t_{n+1/2}),$$
$$k = 1, 2, \ldots, m, \tag{10a}$$

$$x_{k,n+1} = x_{k,n+1/2} + (h/2) f_k(x_{1,n+1/2}, x_{2,n+1/2}, \ldots, x_{k-1,n+1/2}, x_{k,n+1}, \ldots, x_{m,n+1}, t_{n+1}),$$
$$k = m,\ m-1, \ldots, 1. \tag{10b}$$

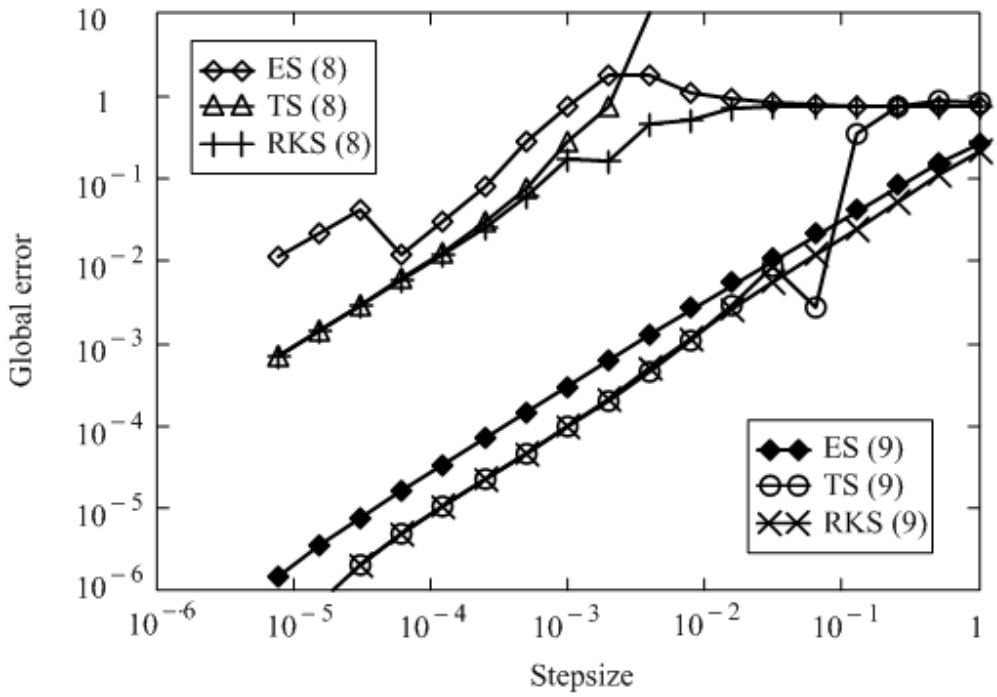

Fig. 3. Global error of methods (2), (7), and (10) applied with various stepsizes *h* to the test problems (8) and (9). The power law exponent at $h \to 0$ for the first four curves is 1 and 1.2 for the last two curves.

5.3. Example 3

To investigate the stability properties of numerical methods applied to oscillatory systems, the scalar harmonic oscillator equation is chosen as a standard test equation

$$y'' = -\omega\, y,\ (\omega > 0), \tag{11}$$

where ω is a real constant [10]. This is the analogue of Dahlquist test equation $y = \lambda\, y$ with $\lambda = \sigma + j\omega$ for a first-order ODE, although the situation is not totally parallel to problems with large negative eigenvalues of the Jacobian matrix. The solutions to (11) are given by the family of sine curves. With initial conditions $y(0) = 1, y' = 0$ the solution is $y = \cos(\omega t)$.

By introducing a new variable $z = y'$, we can rewrite (11) as first-order system

$$y' = f(y,z),\ z' = g(y,z)$$

with purely imaginary eigenvalues ($\pm i\omega$) of the Jacobian matrix.

The asymptotic slope of a global error as a function of stepsize in logarithmic scales is equal to scheme's order only for BDF2 now, as Fig. 3 shows. The effective order of TR-BDF2 is three at the imaginary axis, and four for TR and TR-BDF4. As a

consequence, the method TR-BDF4 is more suited for oscillatory problems than the method TR-BDF2.

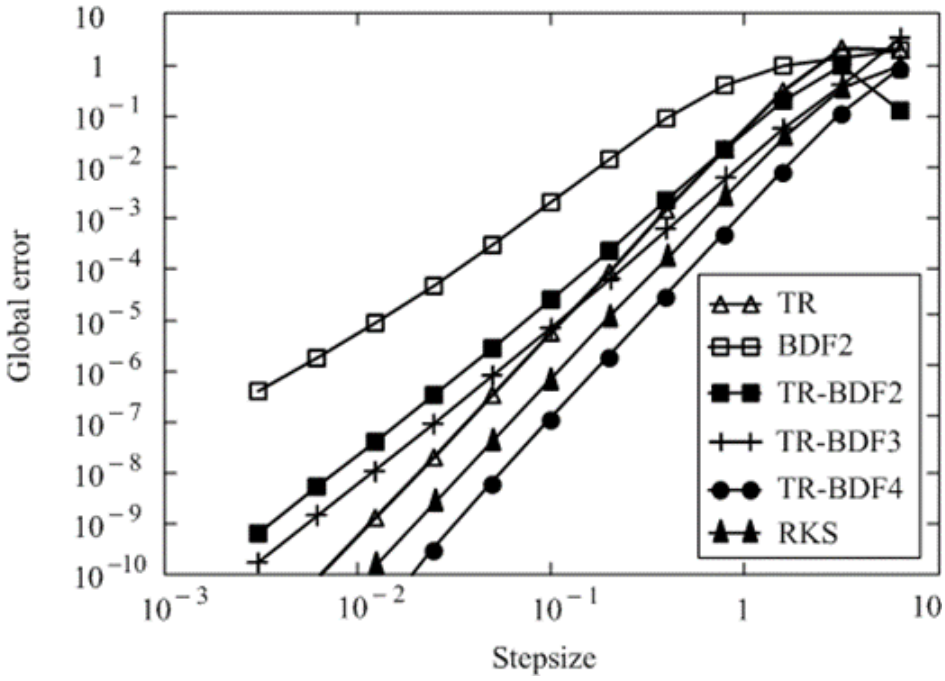

Fig. 4. Global error $\varepsilon = |y_n - y(t_n)|$ at $t_n = 2\pi$ of (1) – (7) methods applied with various stepsizes *h* to the test problem (11). The asymptotically power law exponent is two for BDF2, three – for TR-BDF2 and TR-BDF-3, and four – for TR, TR-BDF-4, and RKS.

The variation of the slope is observed only at the points of local extrema. At any other time, asymptotical slope of all the curves is two as at Fig. 1. In Fig. 2 t_n is equal to the period of oscillation. The effective value of the order at the imaginary axis is determined by the factor of numerical dumping *a* and frequency distortion *b* [10]. The schemes TR and TR-FDN4 have $a = 0$.

6. Disscussion

In this paper we present method of estimation the error of numerical solving of highly oscillation SODE for the first time. Properties of methods of numerical solution of highly oscillatory ordinary differential equations are describe in [10] by two parameters – factor of numerical damping, and relative period error. Here for the various methods comparison only one standard parameter is used – the effective order of the global error in the point of the analytical solutions (10) corresponding to the local maximum. Here we are using the principle *entia non sunt multiplicanda sine*

necessitate.

Method TR-BDF4, as Fig. 5 shows, isn't strictly A(π/2)-stable. As Fig. 4 shows, the error is negligible within the prescribed tolerance $reltol > 10^{-9}$. For (7) $|A(\omega h)| = 1$ with $\omega h < 2$ [6].

Modified L-stable trapezoid methods are known; however, they possess either reduced accuracy [11], or stability [12].

Method (7) is a viable alternative to the conventional techniques for simulating large scale integrated circuits since sufficient conditions for its convergence are quite mild and are satisfied by a large class of practical circuits [5]. We note, however, that the effective order of second-order methods under Seidel iterations, as Fig. 3 shows, decreases somewhat.

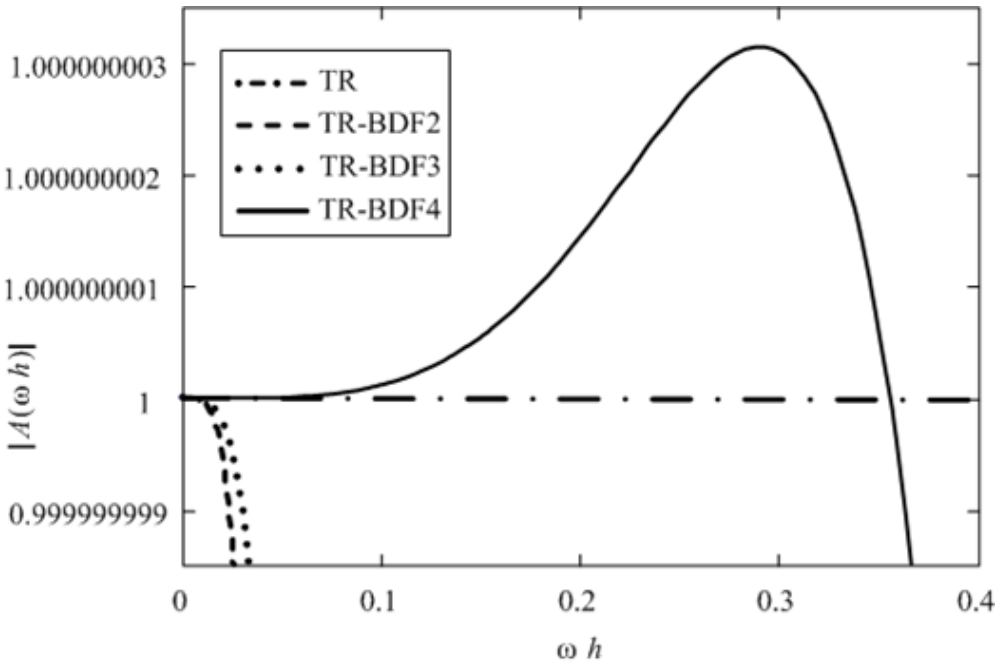

Fig. 5. The modulus of the error function on the imaginary axe. The trapezoid method is strictly A(π/2)-stable only.

The process of developing software from a mathematically specified method is complex: it involves constructing control structures, selecting iterative methods and termination criteria, choosing norms and many more decisions. Two different implementations of the same method may show significant differences in performance [13]. Here we investigated the properties of SODE solving methods.

The engineer is dealing with software codes. The features of the codes should be investigated separately [14].

7. Conclusion

This paper presents a one-step multi-stage backward differentiation formula (TRBDF-4) of a second-order, as a generalization of the TR-BDF composite scheme. This scheme is equivalent to singly diagonally implicit Runge-Kutta (SDIRK) methods of special type, and regarded as the one-step analogs of the multi-step Gear's methods, the so-called backward differentiation formulas. Unlike the standard BDFs and TR-BDF2, however, TR-BDF4 maintains the accuracy of the trapezoidal formula at imaginary axe. TR-BDF4 is trivial to implement in an existing TR-based code with virtually no added computational cost. TR-BDF4 is the best method of the second order for analysis of RF circuits in a time domain as it has L-stability and no damping along a large part of the imaginary axis. TR-BDF4 scheme can serve as a compromise solution to the problem of the oscillatory and simultaneously stiff circuit's analysis, for example, simulation of an oscillator of the harmonic oscillations [15, 16].

Finally, we note that the method of analysis of RF circuits can be called the best strictly if it has no damping throughout the all imaginary axis. However, this requirement holds only with symmetric stability function, i.e., in the absence of L-stability. The absence of damping along the part of imaginary axis has a number of methods. Among them can be noted implicit method using a scheme with combination of Newton and symmetrical Seidel iterations [17] and hybrid method developed on the basis of Rado IIA and Lobatto IIIA methods [18].

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